Original Article

Conservation of angular momentum and virial theorem in variable mass dynamics

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Keywords:
Conservation, momentum, inertia, gravitational, force, moment, angular, limits, dynamics, celestia Virial, mass, inequality, theorem, particles, law, energy, variable, limiting, functions, variable, formula, sharp, maximum, minimum, spacing

Abstract
Whittaker's mass variation law in variable mass dynamics is applied to many- body problem of celestial mechanics with a view to re-establishing thereof some important theorems. It is proved that the conservation of angular momentum still holds irrespective of the manner of mass variation. The virial theorem, in several forms, with the same mass variation principle as Whittaker's has been proposed and proved. Needless to mention that classical Celestial Mechanics is based on a single physical law viz Universal law of gravitation which in the present subject is coupled with a relevant mass-variation law, Sundman's inequality and the law of energy are modified in the light of mass-variation principle in case of a system of n bodies, for which the sum of kinetic and potential energies represents an integral equation with time as an independent variable. A sharp form of virial theorem and hence growth of the system with variable masses are studied and analyzed in comparison with fixed mass criteria [1]. Mathematical analysis including L'Hospital's rule and convolution theorem of integral equation are applied to evaluate some limiting functions involving the moment of inertia, total mass and time and also to set forth some inequality identities involving the initial total mass and the distance of maximum and minimum spacing between the particles (celestial bodies).

1. Introduction

In most of the textbooks of celestial mechanics [2] we have encountered 'conservation of angular momentum' where motion is concerned with particles of constant masses. Whitaker introduced variable-mass dynamics of classical nature and reformulated Lagrange's equations, and Lagrange-Jacobi identities [2],[3] of celestial mechanics assuming a relevant law of mass variation in many-body problem and referred to his formula involving moment of inertia as viral theorem.

The present author [4] studied mass-variation dynamics with generalized law of gravitational force. Basing on Whittaker's principle of mass variation in celestial mechanics, some well-known theorems and formulae have been re-established in the present feature.

Conservation of angular momentum with variable masses: The angular momentum $\vec{A}$ of n particles ($\vec{r}_i$) of masses m_i,($i = 1,2,..,n$) is recalled as

$$\vec{A} = \sum_i \vec{r_i} \times (m_i \dot{\vec{r_i}})$$

Or

$$\frac{d\vec{A}}{dt} = \sum_i \vec{r_i} \times \frac{d}{dt}(m_i \dot{\vec{r_i}}) \tag{1.1}$$

Here we have considered that the masses are varying and as such m_i denotes the mass of the i^{th} particle at time t. The potential function [3] (or energy) for Newton's law of attraction is rewritten as

$$V = -\sum_{i,j} \frac{Gm_i m_j}{r_{ij}} \quad (G = \text{the universal gravitational constant}) \tag{1.2}$$

such that with $\vec{r}_i = (x_i, y_i, z_i)$ the rate of change in momentum being equal to space rate of potential with negative sign,

$$\frac{d}{dt}(m_i \dot{\vec{r_i}}) = -(i\frac{\partial v}{\partial x_i} + j\frac{\partial v}{\partial y_i} + k\frac{\partial v}{\partial z_i}) = -\sum_{x,y,z} \frac{Gm_i m_j}{r_{ij}^2}(\frac{i\partial r_{ij}}{\partial x_i})$$

$$\frac{d}{dt}(m_i \dot{\vec{r_i}}) = -\sum_{x,y,z} \frac{Gm_i m_j}{r_{ij}^2}(\frac{(x_i - x_j)i}{r_{ij}})$$

because $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$ gives $r_{ij}(\frac{\partial r_{ij}}{\partial x_i}) = (x_i - x_j)$.

Substituting (1.3) in (1.1), we get

$$\dot{\vec{A}} = -\sum_i \sum_{j,\, i \neq j} \frac{Gm_i m_j}{r_{ij}^3} \vec{r_i} \times (\vec{r_i} - \vec{r_j}) = \sum_i \sum_j \frac{Gm_i m_j}{r_{ij}^3}(\vec{r_i} \times \vec{r_j} = 0$$

The right hand side of equation (1.4) vanishes because occurrence of each term like ($\vec{r}_i \times \vec{r}_j$) is nullified by occurrence of each term like ($\vec{r}_j \times \vec{r}_i$) so that

$$\sum(\vec{r}_i \times m_i \dot{\vec{r}}_i) = \vec{C} = \text{constant} \tag{1.5}$$

which shows that the angular momentum is also conserved in variable-mass dynamics whatsoever may be the law of mass variation.

2. Modification of Sundman's inequality in variable mass dynamics

Sundman's inequality is an important consequence of the conservation of angular momentum and is found in almost all textbooks [2],[3] of Celestial Mechanics where the masses of the concerned particles are fixed. In this section an attempt has been made to obtain an inequality condition on the same lines considering mass variation of the particles. Because of conservation of angular momentum in variable mass system as well, equation (1.5) can be rewritten as

$$\begin{aligned} C &= |\sum_i (\vec{r_i} \times m_i \vec{v_i})| \\ &\leq \sum m_i |\vec{r_i} \times \vec{v_i}| \\ &\leq \sum m_i |\vec{r_i}||\vec{v_i}| \sin\theta_i \quad (\theta_i \\ &\leq \sum_i (\sqrt{m_i r_i})(\sqrt{m_i v_i}) \end{aligned}$$

is the angle between $\vec{r}_i$ and the velocity $\vec{v}_i$)

Then by Cauchy's inequality,

$$C^2 \leq \sum_i (m_i r_i^2)(m_i v_i^2) = (2\phi)(2T) \tag{2.1}$$

where 2ϕ and T are respectively the moment of inertia and kinetic energy of the system of n particles as defined by Whittaker [1]. Interestingly equation (2.1) is identical with its counterpart in non-variable mass dynamics [2],[3]. In order to derive the present version of Sundman's inequality we need to mention Whittaker's law of mass variation[1] and his moment of inertia 2ϕ-formula[1] (1.3)

$$\frac{d}{dt} m_i(t) = \mu(t) m_i(t) \qquad (i = 1, 2, .., n)$$

$$\frac{d^2}{dt^2}\phi - \frac{d}{dt}(\mu\phi) = 2T + V \qquad (2.2)(2.3)$$

(1.4)

where μ is a function of time t. In the present author's earlier paper [4], the law of energy is modified in the light of variable-mass dynamics as

$$T+V=h-\int_0^t \mu T d\tau \qquad (2.4)$$

where h is the total energy at time t = 0 when the mass variation starts. Combining (2.1),(2.3) and (2.4), the inequality can be re-established basing on Whittaker's mass variation law (2.3) such that

$$C^2<\frac{4\phi}{\mu}\{\dot{T}-\dddot{\phi}+(\mu\ddot{\phi})\}$$

or, in another form,

(The dot sign indicates differentiation with respect to time t)

$$C^2\leq 4\phi\{\ddot{\phi}-(\dot{\mu\phi})-h+\int_0^t \mu T d\tau\}$$

3. Virial theorem with mass variation:

Needless to make any mention of the so-called virial theorem found in many branches of physical science. Here is proposed in the light of Whittaker's variable mass dynamics a viral theorem that stands as a modification of what is done in textbook[2] of Celestial Mechanics.

Statement of the proposed virial theorem: If

(a) Φ and T remain bounded for t >0,

(b) the mass varies according to Whittaker's law $\dot{m}_i=\mu(t)m_i(t)$,

(c) the two limits

$$(i)\hat{T}_\mu=\lim_{t\to\infty}\frac{1}{t}\int_0^t T\{1-\mu(t-\tau)\}d\tau$$

$$(ii)\hat{V}=\lim_{t\to\infty}\frac{1}{t}\int_0^t V(\tau)d\tau,\quad \hat{T}=\lim_{t\to\infty}\frac{1}{t}\int_0^t T(\tau)d\tau \qquad (3.1)(3.2)$$

exist and

$$2\hat{T}=-\hat{V} \qquad (3.3)$$

then $\hat{T}_\mu=-h$ where

$$\hat{T}_\mu=\lim_{t\to\infty}\frac{1}{t}\int_0^t T\{1-\mu(t-\tau)\}d\tau$$

Combining the relationships from (3.1) to (3.3) with (2.4), we have

$$\hat{T}+\hat{V}=\lim_{t\to\infty}\frac{1}{t}\int_0^t (T+V)d\tau=h-\lim_{t\to\infty}\frac{1}{t}\int_0^t\left[\int_0^t \mu T(t')dt'\right]d\tau$$

Therefore, by convolution theorem and transposition,

where $\hat{T}_\mu$ is defined as

$\hat{T}_\mu$=-h

$$\hat{T}_\mu=\lim_{t\to\infty}\frac{1}{t}\int_0^t T\{1-\mu(t-\tau)\}d\tau \qquad (3.5)$$

or otherwise

$$\hat{T}=-h-\hat{\epsilon} \qquad (3.6)$$

$$\hat{\epsilon}=\lim_{t\to\infty}\frac{1}{t}\int_0^t \mu(t-\tau)Td\tau \qquad (3.7)$$

is the additional term due to mass variation.

4. A sharp form of viral theorem in variable mass dynamics

This is established in the case of variable mass dynamics as a counterpart of Pollard's results[2]

Statement of the sharp form of the viral theorem:

$$\hat{T}_\mu=\lim_{t\to\infty}\frac{1}{t}\int_0^t T\{1-\mu(t-\tau)\}d\tau=-h \qquad (4.1)$$

is true if and only if

$(i)\ \lim_{t\to\infty}\dfrac{\phi e^{-\int_0^t \mu d\tau}}{t^2}=0$ or otherwise $\lim_{t\to\infty}\dfrac{\phi}{(Mt^2)}=0$ if $\mu(t)>0$ for all t

$(ii)\ \lim_{t\to\infty}\dfrac{\phi}{(t^2)}=0$ if $\mu(t)<0$ for all t.

Integrating both sides of Whittaker's formula (2.3) and then dividing by t,

$$\lim_{t\to\infty}\left(\frac{\dot{\phi}-\mu\phi}{t}\right)=\lim_{t\to\infty}\left[\frac{2}{t}\int_0^t Tdt+\frac{1}{t}\int_0^t Vdt+\frac{K}{t}\right]$$

$$=\hat{T}+\lim_{t\to\infty}\frac{1}{t}\int_0^t\left[h-\int_0^{t'}\mu T d\tau\right]dt' \quad \text{(Integrating by parts)}$$

$$=h+\lim_{t\to\infty}\frac{1}{t}\int_0^t\left\{1-\mu(t-\tau)\right\}Td\tau$$

$$=h+\hat{T}_\mu=0 \quad (\because (4.1))$$

(Using convolution theorem), K=constant.

Hence, for each $\epsilon>0, (\frac{\dot{\phi}-\mu}{t})<\epsilon\to 0$

After multiplying both sides of the inequality by t and then integrating with respect to t,

$$\phi e^{-\int_0^t \mu d\tau}<\epsilon\int_0^t \tau(e^{-\int_0^\tau \mu dt'})d\tau+A'$$

$$<\epsilon\int_0^t \tau d\tau+A'=\frac{\epsilon t^2}{2}+A'$$

(A' =Constant of integration)

so that

$$\lim_{t\to\infty}\frac{\phi e^{-\int_0^t \mu d\tau}}{t^2}=0$$

, for $\mu(t)$ >0 for all t and ϕ>0. (4.2)

Again by virtue of the mass variation (2.2), we get $\dot{M}=\mu M$ so that

$$M=M_0 e^{\int_0^t \mu d\tau} \qquad (4.3)$$

where M is the total mass of the particles at time t and M_0 the initial total mass.

Substituting (4.3) in (4.2), we get

$$\lim_{t\to\infty}\frac{\phi}{Mt^2}=0 \quad \text{for } \mu(t)>0 \qquad (4.4)$$

Now let us consider the case when $\mu(t)$ <0 for t, i.e. μ' >0 where $\mu'(t)=-\mu(t)$. We have previously

$$\phi e^{-\int_0^t \mu d\tau}<\epsilon\int_0^t \tau(e^{-\int_0^\tau \mu dt'})d\tau+A'$$

Putting $\mu'=-\mu$.

$$\frac{\phi}{t^2}<\frac{\epsilon\int_0^t \tau(e^{\int_0^\tau \mu' d\tau'})d\tau+A'}{t^2 e^{\int_0^t \mu' d\tau}}$$

The integrals in the above inequality are always positive for $\mu'(t)$ >0 and as such tend to ∞ as $t\to\infty$.

Therefore,

$$\lim_{t\to\infty}\frac{\phi}{t^2}\leq\lim_{t\to\infty}\frac{\epsilon\int_0^t \tau(e^{\int_0^\tau \mu' d\tau'})d\tau+A'}{t^2 e^{\int_0^t \mu' d\tau}}$$

(Differentiating the numerator and denominator so as to apply L'Hospital's rule)

$$\leq\lim_{t\to\infty}\frac{\epsilon t e^{\int_0^t \mu' d\tau}}{2te^{\int_0^t \mu' d\tau}+t^2 e^{\int_0^t \mu' d\tau}\mu'}$$

$$=\lim_{t\to\infty}\frac{\epsilon}{2+t\mu'}$$

Finally $\lim_{t\to\infty}\dfrac{\phi}{t^2}=0$ for $\mu(t)$ <0 (4.5)

In the foregoing analysis if we take $\mu(t)$ = 0 ,we shall obtain those concerned with non variable-mass dynamics, as obtained by pollard [2]. The present virial theorem can be utilized by appropriate choice of the function $\mu(t)$. As for the first example let us take $\mu(t)=e^{pt}>0$, where p is a positive constant. By the use of (2.2),

$$M=M_0 e^{\int_0^t e^{pt'}dt'}=M_0 e^{\frac{e^{pt}-1}{p}}$$

where upon equation (4.4) gives

$$\lim_{t\to\infty}\frac{\phi e^{\frac{1-e^{pt}}{p}}}{M_0 t^2}=0 \qquad (p>0)$$

In the second example we take $\mu(t)$ = tan t so that $M=M_0 e^{\int_0^t \tan\tau d\tau}=M_0\sec t$

Then (4.4) becomes

$$\lim_{t\to\infty}\frac{\phi}{M_0 t^2\sec t}=0 \qquad (p>0)$$

The above virial theorem of variable-mass dynamics is valid when $\mu(t)$ is monotonic, i.e. mass of each particle is either increasing or decreasing with time.

t of integration and because(2.4))

5. Growth of the system with variable masses

Let us recall the potential energy function (1.2),

$$V = -\sum_{i,j} \frac{Gm_i m_j}{r_{ij}}$$

where the summation is taken over the indices such that $1 \le i < j \le n$. Suppose $r = Min\{r_{ij}\}$ i.e. $r \le r_{ij}$. Then $-V \le \frac{A}{r}$ where the constant A depends on the masses only. However, under this caption an analysis has been carried out using the potential energy function (1.2), energy equation (2.4), Whittaker's moment of inertia function [1] and differential equation (2.3), to yield some inequality conditions in 'variable-mass dynamics' as counterparts of those given by pollard[2].

Case 1: Consider $h < 0$ and $\mu(t) > 0$ in equation (2.4) whence, because of $T \ge$

$0, -V \ge |h| + \int_0^t \mu T d\tau \ge |h|$. It can be noted that potential energy concerned with motion under Newton's gravitation is of negative sign, i.e. $-V > 0$.

Hence combining the above two inequality conditions,

$$\frac{A}{r} \ge |h| \qquad (5.1)$$

which reveals that if the energy h is negative, the minimum distance is bounded. But the converse is not true.

If m,m' be the two smallest masses at any time t,

$$-V \ge Gmm' \sum \frac{1}{r_{ij}}$$

At a particular instant of time, r becomes one of r_{ij}, and as such the sum on the right side contains $\frac{1}{r}$ giving

$$-V \ge \frac{Gmm'}{r}$$

Combining this with the earlier one,

$$B \le -Vr \le A \qquad (5.2)$$

where A and B are positive functions of the masses which roughly indicate that $-V^{-1}$ is a measure of the minimum spacing between the particles. Let us rewrite Whittaker's [1] moment of inertia function as

$$\phi = \sum \frac{1}{2M} m_i m_j r_{ij}^2, 1 \le i < j \le n$$

If R be the maximum of r_{ij} at some instant

$\phi \le A_1 R^2$

where A_1 depends only on the masses and is an overall function of time t because the masses are varying with time.

Again if m and m' be he two smallest masses at time t,

$$\phi \ge \frac{mm'}{2M} \sum r_{ij}^2$$

Now R being one of r_{ij} at time t

$$\phi \ge \left(\frac{mm'}{2M}\right) R^2 = B_1 R^2$$

where B_1 is a function of masses or is an over all function of time t.

Hence

$$B_1 R^2 \le \phi \le A_1 R^2 \qquad (5.3)$$

which reveals that $\sqrt{\phi}$ is a measure of R, the maximum spacing between the particles. In order to throw some light on how rapidly a system of variable masses can expand dynamically we establish the following relationships.

If $r \ge \delta > 0$, then combining equation (5.2) with (2.4) and Whittaker's equation

(2. 3), we obtain

$$\ddot{\phi} - (\dot{\mu\phi}) = 2h - V - 2\int_0^t \mu T d\tau \le \frac{A}{\delta} + 2h = A''$$

Integrating with respect to t twice,

$$\emptyset e^{-\int_0^t \mu d\tau} \le \int_0^t (A'' t + A''') e^{-\int_0^t \mu d\tau} dt' + A^{iv}$$

$$\emptyset \le (A'' t^2/2 + A''') e^{\int_0^t \mu d\tau} + A^{iv} e^{\int_0^t \mu d\tau} \quad \left(\mu > 0, e^{\int_0^t \mu d\tau} > 1\right) \quad \text{(by (4.3)}$$

$$\emptyset \le \frac{M}{M_0}\left(\frac{A'' t^2}{2} + A''' + A^{iv}\right) \qquad \text{(by (4.3))}$$

This is by use of of the total mass-variation law.

Or, $\emptyset \le M(D_1 t^2 + D_2 t + D_3)$ =MD/M$_0$

where the constants involving the total mass M_0 are given by

$D_1 = \frac{A''}{2M_0}$,

$$D_2 = \frac{A'''}{M_0}, D_3 = \frac{A^{iv}}{M_0}, A^l = constant, l = ', '', ''', iv$$

Combining this inequality with (5.3), we have

$$R \le P\sqrt{M} \qquad (5.4)$$

where $P = \left(\frac{DB_1^{-1}}{M_0}\right)^{\frac{1}{2}}$ = a function of masses and time.

Case 2: Consider $h > 0$ and $\mu > 0$ in equation (2.4). Once again from equations

(2.3) and (2.4), we have

$$\ddot{\phi} - (\dot{\mu\phi}) = 2h - V - 2\int_0^t \mu T d\tau \le \frac{A}{r} + 2h = A^v$$

Integrating twice with respect to t as before,

$\phi \le \frac{ME}{M_0}$ (E is a function of time)

Combining this with (5.3) as before,

$M_0 B_1 R^2 \le ME$ or $R \le Q\sqrt{M}$

where $Q = \left(\frac{EB_1^{-1}}{M_0}\right)^{\frac{1}{2}}$ = a function of masses and time.

Case 3: In this case is taken $\mu < 0$ and $h < 0$. Recalling the energy equation (2.4) of variable-mass dynamics we can write

$$-V = T - h - \int_0^t \mu' T d\tau \qquad (\mu' = -\mu > 0)$$

$\ge K$, K is a positive constant, provided

$\lim_{t\to\infty} \int_0^t \mu' T d\tau$ exists finitely, i.e. $\int_0^t \mu' T d\tau$ is bounded. Then by the use of inequality (5.2),

$A/r \ge \_V \ge K$

or

$$r \le \frac{A}{K} \qquad (5.5)$$

which reveals that if the energy h is negative, the minimum distance is bounded with *the decreasing mass. Here equations (2.3) and (2.4) give*

$\phi^{\cdot\cdot} - (\mu\phi)^{\cdot} = 2h - V + 2\int \mu' T\, d\tau\ 0 > 2h - V > 2h \qquad (-V \ge 0)$

But in view of (5.2) and $r \ge \delta > 0$,

$$\ddot{\phi} - (\dot{\mu\phi}) \le \frac{A}{\delta} + K_1 + 2h$$

because of the boundedness of $-V$ as elucidated in the beginning of this section and because of the fact that $2\int_0^t \mu' T d\tau$ is bounded and $\le K_1$ (constant).

Integrating it twice with respect to t,

$$\phi e^{\int_0^t \mu' d\tau} \le \int_0^t \left(\left(\frac{A}{\delta} + K_1 + 2h\right)t' + c'\right) e^{\int_0^{t'} \mu' d\tau} dt' + c''$$

(Integrating by parts)

$$= \left(\frac{A}{\delta} + K_1 + 2h\right)\left[\frac{t^2}{2} e^{\int_0^t \mu' d\tau} - \int_0^t \frac{t^2}{2} e^{\int_0^t \mu' d\tau} \mu' d\tau\right] + c'\left(t e^{\int_0^t \mu' d\tau} \mu' d\tau - t e^{\int_0^t \mu' d\tau}\right) + c''$$

$$\phi \le \left(\frac{A}{\delta} + K_1 + 2h\right)\frac{t^2}{2} + c' t + c''\left(e^{-\int_0^t \mu' d\tau}\right) \qquad \left(e^{-\int_0^t \mu' d\tau} < 1\right)$$

$\phi \le F(t)$

Combining this with inequaliy (5.3)),

$$R \le (FB_1^{-1})^{\frac{1}{2}} \qquad (5.6)$$

where c', c'' are constants; c'' is a positive constant; $F(t)$ is a nonlinear function of time.

Case 4: Now we assume $h > 0, \mu < 0$, so that in the same way as before, $-\mu = \mu' > 0$

$$\ddot{\phi} - (\dot{\mu\phi}) = 2h - V - +2\int_0^t \mu' T d\tau \ge 2h.$$

Integrating twice with respect to t and with the same technique as before,

$$\phi e^{\int_0^t \mu' d\tau} - \phi_0 \ge 2\int_0^t \left(ht + \frac{a}{2}\right) e^{\int_0^t \mu' d\tau} dt \ge (ht^2 + at) \qquad \left(e^{\int_0^t \mu' d\tau} > 1\right)$$

Because of (4.3)

$$\phi \geq [(ht^2 + at) + \phi_0]\frac{M}{M_0}$$

where $2\phi_0$ is the moment of inertia at time $t = 0$, which, when combined with (5.3)

$$A_1 R^2 \geq \phi \geq [ht^2 + at + \phi_0]\frac{M}{M_0}$$

$$R \geq \left(\frac{hM}{A_1 M_0}\right)^{\frac{1}{2}}\left(t^2 + \frac{at}{h} + \frac{\phi_0}{h}\right)^{\frac{1}{2}} \qquad (5.7)$$

Hence if μ <0 and h >0,R grows at least as fast as the second power of t. With the help of the foregoing theory in mass-variable dynamics, we can set up another form of the viral theorem:

Theorem 5.1 *Prove that*

$$\hat{T}_\mu = \lim_{t\to\infty}\int_0^t \left\{1 - \mu(t-\tau)\right\}\frac{T\tau}{t}d\tau = -h$$

is true in Whittaker's variable-mass dynamics if and only if

lim $t^{-1}R(t) = 0$ *,for* $\mu(t)$ >0 *for all t.* $\lim_{t\to\infty}\frac{\sqrt{M}R(t)}{t} = 0$,for $\mu(t)$ >0 *for all t.*

Proof: We recall the definition of moment of inertia (2ϕ) as

$$\phi = \frac{1}{2}\sum\frac{m_i m_j}{M}r_{ij}^2 \geq \frac{mm'}{2M}\sum r_{ij}^2 \qquad (5.8)$$

where m and m' are the two smallest masses at time t. If R is the maximum of r_{ij}, i.e., the maximum spacing between the particles gives for each $r_{ij} < R$

$$\phi \leq \sum\frac{m_i m_j}{2M}R^2 \qquad (5.9)$$

From(5.8)and((5.9),

$$\frac{mm'}{2M}R^2 \leq \phi \leq (\sum\frac{m_i m_j}{2M})R^2 \qquad (5.10)$$

If m_0,m'_0 and M_0 be masses of the two smallest particles and the total mass of all the particles at time $t = 0$, then by Whittaker's law (2.2), $m = m_0 e^{\int_0^t \mu d\tau}$, $m' = m'_0 e^{\int_0^t \mu d\tau}$ $M = M_0 e^{\int_0^t \mu d\tau}$ so that (5.11) gives

$$\left(\frac{m_0 m'_0}{2M_0}\right)\left(e^{\int_0^t \mu d\tau}\right)R^2 \leq \phi \leq \left(\sum\frac{m_i m_j}{2M_0}e^{\int_0^t \mu d\tau}\right)R^2$$

$$A_2 R^2 \leq \frac{\phi}{(M_0 e^{\int_0^t \mu d\tau})} \leq B_2 R^2$$

where A_2 and B_2 are constants depending on the initial masses.

$$\frac{A_2 R^2}{t^2} \leq \frac{\phi}{Mt^2} \leq \frac{B_2 R^2}{t^2} \qquad (5.11)$$

we have already proved the sharp form (4.3) of the virial theorem of variable mass dynamics:

$\lim_{t\to\infty}\frac{\phi}{Mt^2} = 0$ for $\mu(t)$ >0 for all t which, in view of (5.11), implies lim

$$_{t\to\infty}\frac{R^2}{t^2} = 0$$

or $\lim_{t\to\infty}\frac{R(t)}{t} = 0$ for $\mu(t)$ >0 (5.12)

Now let us consider the case $\mu(t)$ <0; recollecting consequence of equation (5.12) we get

$$A_2(\frac{MR^2}{t^2}) \leq \frac{\phi}{t^2}$$

which, again because of the sharp form (4.5) of the viral theorem of the present design, leads to

$$\lim_{t\to\infty}\frac{MR^2}{t^2} = 0, \text{ or } \lim_{t\to\infty}\frac{\sqrt{M}R(t)}{t} = 0 \qquad (5.13)$$

Finally, we shall establish another result which follows due to the sharp form of the viral theorem of mass-variation:

Let ρ be the largest of the distances $r_1,...,r_n$ of the masses from the origin O. Prove that

$m\rho^2 \leq 2\phi \leq M\rho^2$,

where m is the smallest mass and also that the assertions

$\hat{T}_\mu = -h$ and $\lim_{t\to\infty}\frac{\rho(t)}{t} = 0$

for $\mu(t)$ >0 along with $\lim_{t\to\infty}\frac{\sqrt{M}\rho(t)}{t} = 0$ for $\mu(t) < 0$.

Proof: By definition

0 are equivalent. By definition,

$$\phi = \frac{1}{2}\sum m_i({x_i}^2 + {y_i}^2 + {z_i}^2)$$

where the mass center of the system of masses is kept fixed at the origin O whence follows

$m\rho^2 \leq m_k\rho^2 \leq 2\phi \leq M\rho^2$ (5.14)

where m_k is mass of the particle having the largest distance ρ from the center O. If m_0 and M_0 be the smallest mass and the total mass of the system at time $t = 0$, employing the mass-variation law (2.2) we get

$$m_0 e^{\int_0^t \mu d\tau}\rho^2 \leq 2\phi \leq M_0 e^{\int_0^t \mu d\tau}\rho^2 \text{ or } \frac{m_0}{M_0}\rho^2 \leq \frac{2\phi}{M} \leq \rho^2$$

or

$$\frac{m_0\rho^2}{M_0 t^2} \leq \frac{2\phi}{Mt^2} \leq \frac{\rho^2}{t^2} \qquad (5.15)$$

which, as a consequence of the sharp form (4.4) and (4.5) of the present virial theorem, gives rise to

$$\lim_{t\to\infty}\frac{\rho(t)}{t} = 0, \ for \mu(t) > 0 \ for \ all \ t$$

$$\lim_{t\to\infty}\frac{\sqrt{M}\rho(t)}{t} = 0, \ for \ \mu(t) < 0 \ for \ all \ t$$

keeping in view (4.1). Nevertheless Harry pollard [2] has projected the results of the above kind in form of an exercise where the masses remain fixed.

References

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